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Lie symmetry reductions and exact solutions of some differential–difference equations

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Abstract

In this paper, a class of differential–difference equations (DDEs) are considered for Lie group analysis. With the help of symbolic computation MATHEMATICA, the continuous Lie point symmetry technique is extended to obtain corresponding infinitesimals. Similarity reductions are derived by solving the characteristic equations. Then some exact solutions are presented by using inverse transformations. In addition, starting from concrete realization of the generalized Virasoro type symmetry algebra $[\sigma(f_1), \sigma(f_2)] = \sigma(f_1' f_2 - f_1 f_2')$, many high-dimensional DDEs can be derived. For example, we give out the (2+1)-dimensional Toda lattice, modified Toda lattice and special Toda lattice in a uniform way.

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1. Introduction

In the recent development of the nonlinear science of differential–difference equations (DDEs), the research of symmetry property and the construction of exact solutions become more and more urgent and important. The Lie symmetry group analysis method for continuous differential equations was originally developed by Sophus Lie and is a highly algorithmic method. There have been considerable important developments in this method which include Lie–Bäcklund symmetry, potential symmetry and so on [1, 2]. Usually, with a continuous differential equation, we can study its invariance, symmetry properties and similarity reductions by means of the Lie symmetry method. In [3–5], Levi and Winternitz have extended the continuous Lie symmetry method to solve some DDEs which include (1+1)-dimensional Toda lattice and (2+1)-dimensional Toda lattice. Despite many software packages such as MathLie [6] and DESOLV [7] having been presented to deal with continuous differential equations, special packages for DDEs have not been invented. The main reason is that we must treat every function at different grid points to a new variable. But to some concrete DDEs, we can solve them by the form of human–computer interaction.

In section 2, we consider the following family of (1+1)-dimensional DDEs [8]:

$$\Delta = u_{tt}(n) - \left(a_1 u_t^2(n) + a_2 u_t(n) + a_3\right)\left(g[u(n+1) - u(n)] - g[u(n) - u(n-1)]\right) = 0, \tag{1.1}$$

where $g[u]$ satisfies $g' = a_1 g^2 + a_4 g + a_5$. Clearly this family generalizes some well-studied DDEs. For example, the Toda lattice

$$u_{tt}(n) - \mathrm{e}^{u(n+1)-u(n)} + \mathrm{e}^{u(n)-u(n-1)} = 0 \tag{1.2}$$

corresponds to $a_1 = a_2 = a_5 = 0, a_3 = a_4 = 1, g[u] = \mathrm{e}^u$. The modified Toda lattice

$$u_{tt}(n) - u_t(n)(\mathrm{e}^{u(n+1)-u(n)} - \mathrm{e}^{u(n)-u(n-1)}) = 0 \tag{1.3}$$

corresponds to $a_1 = a_3 = a_5 = 0, a_2 = a_4 = 1, g[u] = \mathrm{e}^u$, the coth-form DDE

$$u_{tt}(n) + u_t^2(n)(\coth[u(n+1) - u(n)] - \coth[u(n) - u(n-1)]) = 0 \tag{1.4}$$

corresponds to $a_2 = a_3 = a_5 4 = 0, a_1 = -a_5 = -1, g[u] = \coth[u]$. The Volterra lattice

$$u_{tt}(n) + u_t^2(n)\left(\frac{1}{u(n+1) - u(n)} - \frac{1}{u(n) - u(n-1)}\right) = 0 \tag{1.5}$$

corresponds to $a_2 = a_3 = a_4 = a_5 = 0, a_1 = -1, g[u] = \frac{1}{u}$.

In [9], the authors have shown that there is an infinite number of equations which have the same Virasoro algebra as the KP equation. Among them only the KP and the cKP are integrable. In section 3, the following (2+1)-dimensional modified Toda lattice [10]

$$u_{xt}(n) - u_x(n)(\mathrm{e}^{u(n+1)-u(n)} - \mathrm{e}^{u(n)-u(n-1)}) = 0. \tag{1.6}$$

and special Toda lattice [11]

$$u_{xt}(n) - [u_x(n+1) + u_x(n)]\,\mathrm{e}^{u(n+1)-u(n)} + [u_x(n) + u_x(n-1)]\,\mathrm{e}^{u(n)-u(n-1)} = 0 \tag{1.7}$$

are considered by Lie symmetry reduction method and we present that these high-dimensional DDEs can be constructed in a uniform way, starting from concrete realization of the generalized Virasoro type symmetry algebra $[\sigma(f_1), \sigma(f_2)] = \sigma(f_1' f_2 - f_1 f_2')$ in section 4. Section 5 is a short summary.

2. Symmetries and exact solutions of equation (1.1)

Knowing the intrinsic Lie symmetry vector field [3–5, 12]

$$V = T(t)\frac{\partial}{\partial t} + \sum_{m\in Z} U(m, t, u(m))\frac{\partial}{\partial u(m)} \tag{2.1}$$

which corresponds to point transformations of the form $\tilde{t} = \Lambda_g(t), \tilde{u}(n) = \Omega_g(n, t, u(n))$, we can obtain the k-order prolongation vector field as follows:

$$\mathrm{pr}^{(\mathrm{k})} V = V + \sum_{m\in Z}\sum_{1\leqslant j\leqslant k} U^{t^j}(m)\frac{\partial}{\partial u_{t^j}(m)}, \tag{2.2}$$

where $U^{t^j}(m) \equiv U^{t^j}(m, t, u(m), u_t(m), \ldots, u_{t^j}(m))$ for simplification and

$$U^{t^j}(m) = D_t U^{t^{j-1}}(m) - (D_t T)u_{t^j}(m). \tag{2.3}$$

Thus the invariance of equation (1.1) needs to calculate

$$U^t(m) = D_t U(m) - (D_t T)u_t(m), \qquad U^{tt}(m) = D_t U^t(m) - (D_t T)u_{tt}(m), \tag{2.4}$$

$$\mathrm{pr}^{(2)} V[\Delta]|_{\Delta=0} = 0. \tag{2.5}$$

Namely, with the help of symbolic computation MATHEMATICA, we have

$$\begin{aligned}
&U_{tt}(n) + 2U_{u(n)t}(n)u_t(n) + U_{u(n)u(n)}(n)u_t^2(n) + (U_{u(n)}(n) - 2T_t)\left(a_1u_t^2(n) + a_2u_t(n) + a_3\right)\\
&\quad\times (g[u(n+1) - u(n)] - g[u(n) - u(n-1)]) - T_{tt}u_t(n) - (2a_1u_t(n) + a_2)\\
&\quad\times (g[u(n+1) - u(n)] - g[u(n) - u(n-1)])(U_t(n) + U_{u(n)}(n)u_t(n) - T_tu_t(n))\\
&\quad- \left(a_1u_t^2(n) + a_2u_t(n) + a_3\right)(a_1g^2[u(n+1) - u(n)] + a_4g[u(n+1) - u(n)] + a_5)\\
&\quad\times (U(n+1) - U(n)) + \left(a_1u_t^2(n) + a_2u_t(n) + a_3\right)(a_1g^2[u(n) - u(n-1)]\\
&\quad+ a_4g[u(n) - u(n-1)] + a_5)(U(n) - U(n-1)) = 0.
\end{aligned} \tag{2.6}$$

Thus, comparing the coefficients of $\left\{u_t^i(n), (i = 2, 1, 0)\right\}$ from the above equation, the following equations

$$\begin{aligned}
u_t^2(n) :\ &U_{u(n)u(n)}(n) - a_1U_{u(n)}(n)(g[u(n+1) - u(n)] - g[u(n) - u(n-1)])\\
&- a_1(a_1g^2[u(n+1) - u(n)] + a_4g[u(n+1) - u(n)] + a_5)(U(n+1) - U(n))\\
&+ a_1(a_1g^2[u(n) - u(n-1)] + a_4g[u(n) - u(n-1)] + a_5)(U(n) - U(n-1)) = 0,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
u_t^1(n) :\ &2U_{u(n)t}(n) - T_{tt} - (T_ta_2 + 2a_1U_t(n))(g[u(n+1) - u(n)] - g[u(n) - u(n-1)])\\
&- a_2(a_1g^2[u(n+1) - u(n)] + a_4g[u(n+1) - u(n)] + a_5)(U(n+1) - U(n))\\
&+ a_2(a_1g^2[u(n) - u(n-1)] + a_4g[u(n) - u(n-1)] + a_5)(U(n) - U(n-1)) = 0,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
u_t^0(n) :\ &U_{tt}(n) + (U_{u(n)}(n)a_3 - 2T_ta_3 - a_2U_t(n))(g[u(n+1) - u(n)] - g[u(n) - u(n-1)])\\
&- a_3(a_1g^2[u(n+1) - u(n)] + a_4g[u(n+1) - u(n)] + a_5)(U(n+1) - U(n))\\
&+ a_3(a_1g^2[u(n) - u(n-1)] + a_4g[u(n) - u(n-1)] + a_5)(U(n) - U(n-1)) = 0
\end{aligned} \tag{2.9}$$

can be derived. Furthermore, comparing the coefficients of $\{g[u(m+1) - u(m)], (m = n, n-1)\}$, we can obtain

$$a_1(U(m+1) - u(m)) = 0, \tag{2.10}$$

$$U_{u(n)}(n) + a_1a_4(U(m+1) - U(m)) = 0, \tag{2.11}$$

$$U_{u(n)u(n)}(n) - a_1a_5(U(n+1) - U(n)) + a_1a_5(U(n) - U(n-1)) = 0, \tag{2.12}$$

$$a_1a_2(U(m+1) - u(m)) = 0, \tag{2.13}$$

$$2a_2U_{u(n)}(n) - 3T_ta_2 - 2a_1U_t(n) - a_2a_4(U(m+1) - U(m)) = 0, \tag{2.14}$$

$$2U_{u(n)t}(n) - T_{tt} - a_2a_5(U(n+1) - U(n)) + a_2a_5(U(n) - U(n-1)) = 0, \tag{2.15}$$

$$a_1a_3(U(m+1) - u(m)) = 0, \tag{2.16}$$

$$a_3U_{u(n)}(n) - 2T_ta_3 - a_2U_t(n) - a_3a_4(U(m+1) - U(m)) = 0, \tag{2.17}$$

$$U_{tt}(n) - a_3a_5(U(n+1) - U(n)) + a_3a_5(U(n) - U(n-1)) = 0, \tag{2.18}$$

with $\{m = n, n-1\}$.

Solving equations (2.10)–(2.18), we can obtain

Case 1. When $a_1 = 0 \Rightarrow g' = a_4g + a_5 \Rightarrow g[u] = c_1\,\mathrm{e}^{a_4u} - \frac{a_5}{a_4}$, equation (1.1) degenerate as

$$u_{tt}(n) - (a_2u_t(n) + a_3)(c_1\,\mathrm{e}^{a_4u(n+1)-a_4u(n)} - c_1\,\mathrm{e}^{a_4u(n)-a_4u(n-1)}) = 0 \tag{2.19}$$

and without loss of generality, we only consider

$$u_{tt}(n) - (a_2 u_t(n) + a_3)(\mathrm{e}^{u(n+1)-u(n)} - \mathrm{e}^{u(n)-u(n-1)}) = 0. \tag{2.20}$$

Furthermore, when $a_2 = 0$, $a_3 = 1$, equation (2.20) just is the (1+1)-dimensional Toda lattice (1.2) and its intrinsic Lie symmetries have been obtained in [3]. Thereby, when $a_2 \neq 0$, the DDE (2.20) has Lie symmetries (infinitesimals)

$$T = c_2 t + c_3, \qquad U = -c_2 n - \frac{c_2 a_3}{a_2} t + c_4 \tag{2.21}$$

and associated with these Lie symmetries, a three-dimensional Lie algebra can be represented by the generators

$$V_1 = t\frac{\partial}{\partial t} - \left(n + \frac{a_3}{a_2} t\right)\frac{\partial}{\partial u(n)}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = \frac{\partial}{\partial u(n)}. \tag{2.22}$$

Similarity reductions can be derived by solving the corresponding characteristic equation $\frac{\mathrm{d}t}{T} = \frac{\mathrm{d}u(n)}{U(n)}$. Namely, when $c_2 \neq 0$, Similarity variable is

$$w(n) = u(t,n) + \frac{a_3}{a_2} t + \left(n - \frac{c_4}{c_2} - \frac{a_3 c_3}{a_2 c_2}\right) \ln[c_2 t + c_3] \tag{2.23}$$

and reduced equation is

$$\frac{-c_2}{a_2} = \mathrm{e}^{w(n+1)-w(n)} - \mathrm{e}^{w(n)-w(n-1)}. \tag{2.24}$$

Thus from an exact solution $w(n) = c_5 + \sum_{j=1}^{n} \ln\left[c_6 - \frac{c_2}{a_2} j\right]$, we can obtain an exact solution of equation (2.20)

$$u(t,n) = \frac{-a_3}{a_2} t + \left(-n + \frac{c_4}{c_2} + \frac{a_3 c_3}{a_2 c_2}\right) \ln[c_2 t + c_3] + c_5 + \sum_{j=1}^{n} \ln\left[c_6 - \frac{c_2}{a_2} j\right]. \tag{2.25}$$

However, when $c_2 = 0$, Similarity variable is

$$w(n) = u(t,n) - \frac{c_4}{c_2} t \tag{2.26}$$

and reduced equation is

$$0 = \mathrm{e}^{w(n+1)-w(n)} - \mathrm{e}^{w(n)-w(n-1)}. \tag{2.27}$$

Thus from an exact solution $w(n) = c_7 n + c_8$, we can obtain another exact solution of equation (2.20)

$$u(t,n) = \frac{c_4}{c_3} t + c_7 n + c_8. \tag{2.28}$$

Case 2. When $a_1 \neq 0$, $a_3 = \frac{a_2^2}{4a_1}$, equation (1.1) degenerate as

$$u_{tt}(n) - \left(a_1 u_t^2(n) + a_2 u_t(n) + \frac{a_2^2}{4a_1}\right)(g[u(n+1) - a_4 u(n)] - g[u(n) - a_4 u(n-1)]) = 0. \tag{2.29}$$

This DDE has Lie symmetries (infinitesimals)

$$T = c_1 t + c_2, \qquad U = \frac{-a_2 c_1}{2a_1} t + c_3. \tag{2.30}$$

When $c_1 \neq 0$, Similarity variable is

$$w(n) = u(t,n) + \frac{a_2}{2a_1} t - \left(\frac{a_2 c_2}{2a_1 c_1} + \frac{c_3}{c_1}\right) \ln[c_1 t + c_2] \tag{2.31}$$

and reduced equation is

$$\frac{-c_1}{\frac{a_2c_2}{2}+a_1c_3} = g[w(n+1)-w(n)] - g[w(n)-w(n-1)]. \tag{2.32}$$

Thus from an exact solution $w(n) = c_4 + \sum_{j=1}^{n} g^{-1}\left[c_5 - \frac{c_1}{\frac{a_2c_2}{2}+a_1c_3} j\right]$, we can obtain an exact solution of equation (2.29)

$$u(t,n) = \frac{-a_2}{2a_1}t + \left(\frac{a_2c_2}{2a_1c_1} + \frac{c_3}{c_1}\right)\ln[c_1t+c_2] + c_4 + \sum_{j=1}^{n} g^{-1}\left[c_5 - \frac{c_1}{\frac{a_2c_2}{2}+a_1c_3} j\right], \tag{2.33}$$

where g^{-1} is defined an inverse operator. To some concrete examples such as equation (1.4) and equation (1.5), we can obtain exact solutions

$$u(t,n) = \frac{c_3}{c_1}\ln[c_1t+c_2] + c_4 + \sum_{j=1}^{n} \mathrm{Arccoth}\left[c_5 + \frac{c_1}{c_3} j\right], \tag{2.34}$$

$$u(t,n) = \frac{c_3}{c_1}\ln[c_1t+c_2] + c_4 + \sum_{j=1}^{n} \frac{1}{c_5 + \frac{c_1}{c_3} j}. \tag{2.35}$$

In addition, obviously, equation (1.1) has a Lie symmetry $T = c_2, U = c_3$, namely, equation (1.1) which include equation (2.29) consequently has an exact solution (2.28).

3. Symmetries and exact solutions of equations (1.6) and (1.7)

To the (2+1)-dimensional modified Toda lattice(1.6) and special Toda lattice (1.7), we have no knowledge of their Lie symmetries. So in the following, we apply the Lie symmetry reduction method [3–5, 12]

$$\mathrm{pr}^k V = V + \sum_{m\in Z} \sum_{1\leqslant i+j\leqslant k} U^{x^it^j}(m)\partial_{u_{x^it^j}(m)}, \tag{3.1}$$

$$U^{x^it^j}(m) = D_x U^{x^{i-1}t^j}(m) - (D_xX)u_{x^it^j}(m) - (D_xT)u_{x^{i-1}t^{j+1}}(m), \tag{3.2}$$

$$= D_t U^{x^it^{j-1}}(m) - (D_tX)u_{x^{i+1}t^{j-1}}(m) - (D_tT)u_{x^it^j}(m) \tag{3.3}$$

directly to them, where we restrict the vector field V as the simplest 'intrinsic' form $V = T(x,t)\partial_t + X(x,t)\partial_x + \sum_{m\in Z} U(x,m,t,u(m))\partial_{u(m)}$. Thus, the equable infinite dimensional Lie algebra with basis,

$$K_1(f) = -f(t)\partial_t + f'(t)n\partial_{u(n)}, \qquad K_2(g) = -g(x)\partial_x, \qquad K_3(h) = -h(t)\partial_{u(n)}, \tag{3.4}$$

can be derived. For simplification, we omit the process. We can reduce the equation by solving characteristic equations,

$$\frac{\mathrm{d}t}{-f(t)} = \frac{\mathrm{d}x}{-g(x)} = \frac{\mathrm{d}u(n)}{f'(t)n - h(t)}. \tag{3.5}$$

Case 1. When $f(t)$ is a constant, we can set $f(t) = 1$ without loss of generality. Then the similarity variables are

$$y = t - \int^x \frac{\mathrm{d}\tilde{x}}{g(\tilde{x})}, \qquad F(y,n) = u(x,t,n) - \int^t h(\tilde{t})\,\mathrm{d}\tilde{t}. \tag{3.6}$$

Case 2. When $f(t)$ is not a constant, we can obtain

$$y = \int^t \frac{\mathrm{d}\tilde{t}}{f(\tilde{t})} - \int^x \frac{\mathrm{d}\tilde{x}}{g(\tilde{x})}, \qquad F(y,n) = u(x,t,n) + n\ln[f(t)] - \int^t \frac{h(\tilde{t})}{f(\tilde{t})}\,\mathrm{d}\tilde{t}. \tag{3.7}$$

From case 1 or case 2, we can obtain the following reduced equations,

$$F_{yy}(n) - F_y(n)(\mathrm{e}^{F(n+1)-F(n)} - \mathrm{e}^{F(n)-F(n-1)}) = 0, \tag{3.8}$$

$$F_{yy}(n) - \mathrm{e}^{F(n+1)-F(n)}[F(n+1)+F(n)]_y + \mathrm{e}^{F(n)-F(n-1)}[F(n)+F(n-1)]_y = 0. \tag{3.9}$$

To the above (1+1)-dimensional modified Toda lattice (3.8) and special Toda lattice (3.9), we can also obtain the following equable Lie algebra with basis,

$$K_4 = y\partial_y - n\partial_{F(n)}, \qquad K_5 = \partial_y, \qquad K_6 = \partial_{F(n)}. \tag{3.10}$$

So the corresponding similarity variables can be obtained from the following characteristic equation,

$$\frac{\mathrm{d}y}{c_3 y + c_4} = \frac{\mathrm{d}F(n)}{-c_3 n + c_5}. \tag{3.11}$$

Case I. When $c_3 = 0$, we can obtain similarity variable $G(n) = F(y,n) - \frac{c_5}{c_4}y$ and reduce equation (3.8) or equation (3.9) to

$$\mathrm{e}^{G(n+1)-G(n)} - \mathrm{e}^{G(n)-G(n-1)} = 0. \tag{3.12}$$

Thus, from an exact solution $G(n) = c_6 n + c_7$ of the above equation, equations (3.6) and (3.7), we can construct two new exact solutions of the modified Toda lattice (1.6) and special Toda lattice (1.7),

$$u(x,t,n) = \int^t h(\tilde{t})\,\mathrm{d}\tilde{t} + \frac{c_5}{c_4}\left(t - \int^x \frac{\mathrm{d}\tilde{x}}{g(\tilde{x})}\right) + c_6 n + c_7, \tag{3.13}$$

$$u(x,t,n) = -n\ln[f(t)] + \int^t \frac{h(\tilde{t})}{f(\tilde{t})}\,\mathrm{d}\tilde{t} + \frac{c_5}{c_4}\left(\int^t \frac{\mathrm{d}\tilde{t}}{f(\tilde{t})} - \int^x \frac{\mathrm{d}\tilde{x}}{g(\tilde{x})}\right) + c_6 n + c_7. \tag{3.14}$$

Case II. When $c_3 \neq 0$, we can obtain similarity variable $G(n) = F(y,n) - \frac{-c_3 n + c_5}{c_3}\ln[c_3 y + c_4]$ and reduce equation (3.9) to

$$\mathrm{e}^{G(n+1)-G(n)} - \mathrm{e}^{G(n)-G(n-1)} = -c_3. \tag{3.15}$$

Thus, from an exact solution $G(n) = c_8 + \sum_{j=1}^n \ln[-c_3 j + c_9]$ of the above equation, equations (3.6) and (3.7), we can construct two new exact solutions of the (2+1)-dimensional modified Toda lattice (1.7),

$$u(x,t,n) = \int^t h(\tilde{t})\,\mathrm{d}\tilde{t} + \frac{-c_3 n + c_5}{c_3}\ln\left[c_3\left(t - \int^x \frac{\mathrm{d}\tilde{x}}{g(\tilde{x})}\right) + c_4\right] + c_8 + \sum_{j=1}^n \ln[-c_3 j + c_9], \tag{3.16}$$

$$u(x,t,n) = -n\ln[f(t)] + \int^t \frac{h(\tilde{t})}{f(\tilde{t})}\,\mathrm{d}\tilde{t} + \frac{-c_3 n + c_5}{c_3}\ln\left[c_3\left(\int^t \frac{\mathrm{d}\tilde{t}}{f(\tilde{t})} - \int^x \frac{\mathrm{d}\tilde{x}}{g(\tilde{x})}\right) + c_4\right] + c_8 + \sum_{j=1}^n \ln[-c_3 j + c_9]. \tag{3.17}$$

In this case, equation (3.9) also can be reduced to

$$e^{G(n+1)-G(n)}(-2c_3n+2c_5-c_3)-e^{G(n)-G(n-1)}(2c_3n+2c_5+c_3)=(c_3n-c_5)c_3. \tag{3.18}$$

Thus, from an exact solution $G(n)=c_{10}+\sum_{j=1}^{n}\ln\left[\frac{c_3+2c_5}{8}-\frac{c_3}{4}j\right]$ of the above equation, equations (3.6) and (3.7), we can construct two exact solutions of equation (1.7),

$$u(x,t,n)=\int^t h(\tilde{t})\,d\tilde{t}+\frac{-c_3n+c_5}{c_3}\ln\left[c_3\left(t-\int^x\frac{d\tilde{x}}{g(\tilde{x})}\right)+c_4\right] \\ +c_{10}+\sum_{j=1}^{n}\ln\left[\frac{c_3+2c_5}{8}-\frac{c_3}{4}j\right], \tag{3.19}$$

$$u(x,t,n)=-n\ln[f(t)]+\int^t\frac{h(\tilde{t})}{f(\tilde{t})}\,d\tilde{t}+\frac{-c_3n+c_5}{c_3}\ln\left[c_3\left(\int^t\frac{d\tilde{t}}{f(\tilde{t})}-\int^x\frac{d\tilde{x}}{g(\tilde{x})}\right)+c_4\right] \\ +c_{10}+\sum_{j=1}^{n}\ln\left[\frac{c_3+2c_5}{8}-\frac{c_3}{4}j\right]. \tag{3.20}$$

4. Generalized Virasoro symmetry subalgebra of some DDEs

In [9], a general class of fourth order scalar partial differential equations(PDEs), invariant under the same group of local point transformations as the KP equation, is obtained. This is a very important reverse direction technology to seek for some significative PDEs. In the following, we extend this method to deal with DDEs. Firstly, we need to realize the Virasoro subalgebra $[\sigma(f_1),\sigma(f_2)]=\sigma(f_1'f_2-f_1f_2')$ in terms of vector fields on the space $S_1\otimes S_2$ of independent and dependent variables, here S_1 is the three-dimensional coordinates (x,m,t) and S_2 is the function $u(x,m,t)$. In order to obtain some concrete significant results, we restrict the vector field V as

$$V=-f(t)\partial_t+0\cdot\partial_x+\sum_{m\in Z}f'(t)m\partial_{u(m)}, \tag{4.1}$$

where $u(m)=u(x,m,t)$ for convenience. In order to construct invariant kth-order differential–difference equations, we need the kth prolongation of the corresponding vector field (3.1)–(3.3). Thus, we have

$$U^{x^i}(m)=0,\qquad (i=1,2,3,\ldots), \tag{4.2}$$

$$U^{t^j}(m)=f^{(j+1)}m+\sum_{h=0}^{j-1}\binom{j}{h}f^{(j-h)}u_{t^{h+1}}(m),\qquad (j=1,2,3,\ldots), \tag{4.3}$$

$$U^{x^it^j}(m)=\sum_{h=0}^{j-1}\binom{j}{h}f^{(j-h)}u_{x^it^{h+1}}(m),\qquad (i,j=1,2,3,\ldots), \tag{4.4}$$

where let $f=f(t)$ for simplification.

To get the explicit elementary invariants of V, we have to solve the corresponding characteristic equations

$$\frac{dt}{-f}=\frac{dx}{0}=\frac{du(m)}{f'm}=\frac{du_t(m)}{f''m+f'u_t(m)}=\cdots. \tag{4.5}$$

After finishing detailed calculations, we can obtain various group invariants, for example,

$$I_{0,0}(m) = \mathrm{e}^{u(m)} f^m, \tag{4.6}$$

$$I_{i,0}(m) = u_{x^i}(m), \tag{4.7}$$

$$I_{0,1}(m) = u_t(m) f + f' m, \tag{4.8}$$

$$I_{i,1}(m) = u_{x^i t}(m) f, \tag{4.9}$$

$$I_{0,2}(m) = u_{tt}(m) f^2 + m f f'' - f'^2 m, \qquad \cdots \tag{4.10}$$

where integer $m \in (n - a, n + b)$ and $i = 1, 2, 3, \ldots$. The general V invariant equation then can be written as follows,

$$\mathrm{Eq}(I_{0,0}(m), I_{i,0}(m), I_{0,j}(m), \ldots) = 0. \tag{4.11}$$

Usually, the invariants $I_{ij}(m)$ are f-dependent. However, to find the general Virasoro type subalgebra integrable equations, we should select out the f-independent equations from equation (4.11). Here we only list the (2+1)-dimensional Toda lattice, modified Toda lattice and special Toda lattice.

Toda lattice

From the following V invariant equation

$$I_{1,1}(n) - \frac{I_{0,0}(n)}{I_{0,0}(n-1)} + \frac{I_{0,0}(n+1)}{I_{0,0}(n)} = 0, \tag{4.12}$$

we can obtain the (2+1)-dimensional Toda lattice [3–5]

$$u_{xt}(n) - \mathrm{e}^{u(n)-u(n-1)} + \mathrm{e}^{u(n+1)-u(n)} = 0. \tag{4.13}$$

Modified Toda lattice

From the following V invariant equation

$$I_{1,1}(n) - I_{1,0}(n)\left(\frac{I_{0,0}(n+1)}{I_{0,0}(n)} + \frac{I_{0,0}(n)}{I_{0,0}(n-1)}\right) = 0, \tag{4.14}$$

we can obtain the (2+1)-dimensional modified Toda lattice (1.6).

Special Toda lattice

From the following V invariant equation

$$I_{1,1}(n) - \frac{I_{0,0}(n+1)}{I_{0,0}(n)}[I_{1,0}(n+1) + I_{1,0}(n)] + \frac{I_{0,0}(n)}{I_{0,0}(n-1)}[I_{1,0}(n) + I_{1,0}(n-1)] = 0, \tag{4.15}$$

the (2+1)-dimensional special Toda lattice (1.7) can be derived. Here, we give out the (2+1)-dimensional Toda lattice, modified Toda lattice and special Toda lattice in a uniform way. In fact, many higher order DDEs can be obtained by using this method.

5. Summary

In this work, we have studied systematically the intrinsic Lie point symmetries [3–5], similarity reductions and exact solutions of some DDEs. We believe that applying this method to solve other DDEs is worth studying. In addition, non-intrinsic Lie symmetries

$$V = T(t, u_i : i \in Z)\frac{\partial}{\partial t} + \sum_{n \in Z} U_n(t, u_i : i \in Z)\frac{\partial}{\partial u(n)} \tag{5.1}$$

and conditional symmetries [3–5] are our next challenge for studying these DDEs. How does one seek for new integrable DDEs from a concrete realization of the generalized Virasoro type symmetry algebra $[\sigma(f_1), \sigma(f_2)] = \sigma(f_1' f_2 - f_1 f_2')$? Furthermore, it will be a very important work to study out a universal procedure to calculate Lie symmetries for DDEs.

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